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## Dade's ordinary conjecture implies the Alperin–McKay conjecture

RADHA KESSAR AND MARKUS LINCKELMANN

**Abstract.** We show that Dade's ordinary conjecture implies the Alperin–McKay conjecture. We remark that some of the methods can be used to identify a canonical height zero character in a nilpotent block.

**Mathematics Subject Classification.** 20C20.

**Keywords.** Dade's ordinary conjecture, Alperin–McKay conjecture, Height zero characters.

Dade proved in [4] that his projective conjecture [4, 15.5] implies the Alperin–McKay conjecture. Navarro showed in [11, Theorem 9.27] that the group version of Dade's ordinary conjecture implies the McKay conjecture. We show here that Dade's ordinary conjecture [3, 6.3] implies the Alperin–McKay conjecture. Let  $p$  be a prime number.

**Theorem 1.** *If Dade's ordinary conjecture holds for all  $p$ -blocks of finite groups, then the Alperin–McKay conjecture holds for all  $p$ -blocks of finite groups.*

The proof combines arguments from Sambale [17] and formal properties of chains of subgroups in fusion systems from [7]. Let  $(K, \mathcal{O}, k)$  be a  $p$ -modular system. We assume that  $k$  is algebraically closed, and let  $\bar{K}$  be an algebraic closure of  $K$ . By a character of a finite group, we will mean a  $\bar{K}$ -valued character. For a finite group  $G$  and a block  $B$  of  $\mathcal{O}G$ , let  $\text{Irr}(B)$  denote the set of irreducible characters of  $G$  in the block  $B$ , and let  $\text{Irr}_0(B)$  denote the set of irreducible height zero characters of  $G$  in  $B$ . For a central  $p$ -subgroup  $Z$  of  $G$  and a character  $\eta$  of  $Z$ , let  $\text{Irr}_0(B|\eta)$  denote the subset of  $\text{Irr}_0(B)$  consisting of

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those height zero characters which cover the character  $\eta$ . The following lemma is implicit in [17].

**Lemma 2.** *Let  $P$  be a finite  $p$ -group, let  $\mathcal{F}$  be a saturated fusion system on  $P$ , and let  $Z \leq Z(\mathcal{F})$ . Suppose that  $\eta$  is a linear character of  $P$ . There exists a linear character  $\hat{\eta}$  of  $P$  such that  $\hat{\eta}|_Z = \eta|_Z$  and  $\text{foc}(\mathcal{F}) \leq \text{Ker}(\hat{\eta})$ .*

*Proof.* First consider the case that  $\eta|_Z$  is faithful. Then  $Z \cap [P, P] = 1$ . Hence by [5, Lemma 4.3],  $\text{foc}(\mathcal{F}) \cap Z = 1$ . The result is now immediate. Now suppose  $Z_0 = \text{Ker}(\eta|_Z)$  and let  $\bar{\mathcal{F}} = \mathcal{F}/Z_0$ . By the previous argument, applied to  $P/Z_0$  and  $\bar{\mathcal{F}}$ , there exists a character  $\hat{\eta}$  of  $P/Z_0$  such that  $\hat{\eta}|_{Z/Z_0} = \eta|_{Z/Z_0}$  and  $\text{foc}(\bar{\mathcal{F}}) \leq \text{Ker}(\hat{\eta})$ . Denote also by  $\hat{\eta}$  the inflation of  $\hat{\eta}$  to  $P$ . Then  $\hat{\eta}$  has the required properties since  $\text{foc}(\bar{\mathcal{F}}) = \text{foc}(\mathcal{F})Z_0/Z_0$ .  $\square$

The following result is a special case of a result due to Murai; we include a proof for convenience.

**Lemma 3** (cf. [9, Theorem 4.4]). *Let  $G$  be a finite group,  $B$  be a block of  $\mathcal{O}G$ , and  $P$  a defect group of  $B$ . Let  $Z$  be a central  $p$ -subgroup of  $G$  and let  $\eta$  be an irreducible character of  $Z$  such that  $\text{Irr}_0(B|\eta) \neq \emptyset$ . Then  $\eta$  extends to  $P$ .*

*Proof.* By replacing  $K$  by a suitable finite extension, we may assume that  $K$  is a splitting field for all subgroups of  $G$ . Let  $i \in B^P$  be a source idempotent of  $B$  and let  $V$  be a  $KG$ -module affording an element of  $\text{Irr}_0(B|\eta)$ . Then  $n := \dim_K(iV)$  is prime to  $p$  (see [13]). Since  $i$  commutes with  $P$ ,  $iV$  is a  $KP$ -module via  $x \cdot iv = ixv$ , where  $x \in P, v \in V$ . Let  $\rho : P \rightarrow \text{GL}_n(K)$  be a corresponding representation and let  $\delta : P \rightarrow K^\times$  be the determinantal character of  $\rho$ . Then  $\delta|_Z = \eta^n$ . The result follows since  $n$  is prime to  $p$ .  $\square$

**Lemma 4.** *Let  $G$  be a finite group, let  $B$  be a block of  $\mathcal{O}G$  with a defect group  $P$ , and let  $Z$  be a central  $p$ -subgroup of  $G$ . Then  $|\text{Irr}_0(B)|$  equals the product of  $|\text{Irr}_0(B|1_Z)|$  with the number of distinct linear characters  $\eta$  of  $Z$  which extend to  $P$ .*

*Proof.* Let  $\mathcal{F} = \mathcal{F}_{(P, e_P)}(G, B)$  be the fusion system of  $B$  with respect to a maximal  $B$ -Brauer pair  $(P, e_P)$ , and let  $\eta$  be a linear character of  $Z$  which extends to  $P$ . Since  $Z \leq Z(\mathcal{F})$ , by Lemma 2 there exists a linear character  $\hat{\eta}$  of  $P$  such that  $\hat{\eta}|_Z = \eta$  and  $\text{foc}(\mathcal{F}) \leq \text{Ker}(\hat{\eta})$ . By the properties of the Broué-Puig  $*$ -construction [1, 16] the map  $\chi \mapsto \hat{\eta} * \chi$  is a bijection between  $\text{Irr}_0(B|1_Z)$  and  $\text{Irr}_0(B|\eta)$ . The result follows by Lemma 3.  $\square$

Slightly strengthening the terminology in [10], we say that a pair  $(G, B)$  consisting of a finite group  $G$  and a block  $B$  of  $\mathcal{O}G$  is a *minimal counterexample to the Alperin–McKay conjecture* if  $B$  is a counterexample to the Alperin–McKay conjecture and if  $G$  is such that first  $|G : Z(G)|$  is smallest possible and then  $|G|$  is smallest possible.

**Proposition 5.** *Let  $(G, B)$  be a minimal counterexample to the Alperin–McKay conjecture. Then  $O_p(G) = 1$ .*

*Proof.* By a result of Murai [10], we have that  $Z := O_p(G)$  is central in  $G$ . Let  $P$  be a defect group of  $B$  and let  $C$  be the block of  $\mathcal{O}N_G(P)$  in Brauer correspondence with  $B$ . By Lemma 4,  $|\text{Irr}_0(B)| = |\text{Irr}_0(C)|$  if and only if  $|\text{Irr}_0(\bar{B})| = |\text{Irr}_0(\bar{C})|$  where  $\bar{B}$  (resp.  $\bar{C}$ ) is the block of  $\mathcal{O}G/Z$  (resp.  $\mathcal{O}N_G(P)/Z$ ) dominated by  $B$  (resp.  $C$ ). The result follows since  $N_{G/Z}(P/Z) = N_G(P)/Z$  and  $\bar{B}$  and  $\bar{C}$  are in Brauer correspondence.  $\square$

Let  $\mathcal{F}$  be a saturated fusion system on a finite  $p$ -group  $P$ , and let  $\mathcal{C}$  be a full subcategory of  $\mathcal{F}$  which is upwardly closed; that is, if  $Q, R$  are subgroups of  $P$  such that  $Q$  belongs to  $\mathcal{C}$  and if  $\text{Hom}_{\mathcal{F}}(Q, R)$  is nonempty, then also  $R$  belongs to  $\mathcal{C}$ . Drawing upon notation and facts from [7, §5],  $S_{\triangleleft}(\mathcal{C})$  is the category having as objects nonempty chains  $\sigma = Q_0 < Q_1 < \cdots < Q_m$  of subgroups  $Q_i$  of  $P$  belonging to  $\mathcal{C}$  such that  $m \geq 0$  and  $Q_i$  is normal in  $Q_m$ , for  $0 \leq i \leq m$ . Morphisms in  $S_{\triangleleft}(\mathcal{C})$  are given by certain ‘obvious’ commutative diagrams of morphisms in  $\mathcal{F}$ ; see [7, 2.1, 4.1] for details. With this notation, the *length* of a chain  $\sigma$  in  $S_{\triangleleft}(\mathcal{C})$  is the integer  $|\sigma| = m$ . The chain  $\sigma$  is called *fully normalised* if  $Q_0$  is fully  $\mathcal{F}$ -normalised and if either  $m = 0$  or the chain  $\sigma_{\geq 1} = Q_1 < Q_2 < \cdots < Q_m$  is fully  $N_{\mathcal{F}}(Q_0)$ -normalised. Every chain in  $S_{\triangleleft}(\mathcal{C})$  is isomorphic (in the category  $S_{\triangleleft}(\mathcal{C})$ ) to a fully normalised chain. There is an involution  $n$  on the set of fully normalised chains which fixes the chain of length zero  $P$  and which sends any other fully normalised chain  $\sigma$  to a fully normalised chain  $n(\sigma)$  of length  $|\sigma| \pm 1$ . This involution is defined as follows. If  $\sigma = P$ , then set  $n(\sigma) = \sigma$ . If  $\sigma = Q_0 < Q_1 < \cdots < Q_m$  is a fully normalised chain different from  $P$  such that  $Q_m = N_P(\sigma)$ , then define  $\sigma$  by removing the last term  $Q_m$ ; if  $Q_m < N_P(\sigma)$ , then define  $\sigma$  by adding  $N_P(\sigma)$  as last term to the chain. Then  $n(\sigma)$  is fully normalised, and  $n(n(\sigma)) = \sigma$ . Denote by  $[S_{\triangleleft}(\mathcal{C})]$  the partially ordered set of isomorphism classes of chains in  $S_{\triangleleft}(\mathcal{C})$ , and for each chain  $\sigma$  by  $[\sigma]$  its isomorphism class. We have a partition

$$[S_{\triangleleft}(\mathcal{C})] = \{[P]\} \cup \mathcal{B} \cup n(\mathcal{B}),$$

where  $\mathcal{B}$  is the set of isomorphism classes of fully normalised chains  $\sigma$  satisfying  $|n(\sigma)| = |\sigma| + 1$ . The following Lemma is a very special case of a functor cohomological statement [7, Theorem 5.11].

**Lemma 6.** *With the notation above, let  $f : [S_{\triangleleft}(\mathcal{C})] \rightarrow \mathbb{Z}$  be a function on the set of isomorphism classes of chains in  $S_{\triangleleft}(\mathcal{C})$  satisfying  $f([\sigma]) = f([n(\sigma)])$  for any fully normalised chain  $\sigma$  in  $S_{\triangleleft}(\mathcal{C})$ . Then*

$$\sum_{[\sigma] \in [S_{\triangleleft}(\mathcal{C})]} (-1)^{|\sigma|} f([\sigma]) = f([P]).$$

*Proof.* The hypothesis on  $f$  implies that the contributions from chains in  $\mathcal{B}$  cancel those from chains in  $n(\mathcal{B})$ , whence the result.  $\square$

**Proposition 7.** *Let  $G$  be a finite group such that  $O_p(G) = 1$ , and let  $B$  be a block of  $\mathcal{O}G$  with nontrivial defect groups. Suppose that Dade’s ordinary conjecture holds for  $B$  and that the Alperin–McKay conjecture holds for any block of any proper subgroup of  $G$ . Then the Alperin–McKay conjecture holds for the block  $B$ .*

*Proof.* Let  $(P, e)$  be a maximal  $B$ -Brauer pair, and denote by  $\mathcal{F}$  the associated fusion system on  $P$ . For  $d$  a positive integer, denote by  $\mathbf{k}_d(G, B)$  the number of ordinary irreducible characters in  $B$  of defect  $d$ . If  $p^d = |P|$ , then  $\mathbf{k}_d(G, B)$  is the number of height zero characters, and if  $p^d > |P|$ , then  $\mathbf{k}_d(G, B) = 0$ .

Let  $\mathcal{C}$  be the full subcategory of  $\mathcal{F}$  consisting of all nontrivial subgroups of  $P$ . We briefly describe the standard translation process between chains in a fusion system of a block and the associated chains of Brauer pairs. The map sending a chain  $\sigma = Q_0 < Q_1 < \cdots < Q_m$  in  $S_{\triangleleft}(\mathcal{C})$  to the unique chain of nontrivial  $B$ -Brauer pairs  $\tau = (Q_0, e_0) < (Q_1, e_1) < \cdots < (Q_m, e_m)$  contained in  $(P, e)$  induces a bijection between isomorphism classes of chains in  $S_{\triangleleft}(\mathcal{C})$  and the set of  $G$ -conjugacy classes of normal chains of nontrivial  $B$ -Brauer pairs (cf. [7, 2.5]). If  $\sigma$  is fully normalised, then the corresponding chain of Brauer pairs  $\tau = (Q_0, e_0) < (Q_1, e_1) < \cdots < (Q_m, e_m)$  has the property that  $e_\tau = e_m$  remains a block of  $N_G(\tau)$ , and by [7, 5.14],  $N_P(\sigma) = N_P(\tau)$  is a defect group of  $e_\tau$  as a block of  $N_G(\tau)$ . Denote by  $n(\tau)$  the chain of Brauer pairs corresponding to  $n(\sigma)$ .

Let  $d > 0$  such that  $p^d = |P|$ . Define a function  $f$  on  $S_{\triangleleft}(\mathcal{C})$  by setting

$$f([\sigma]) = \mathbf{k}_d(N_G(\tau), e_\tau)$$

for any fully normalised chain  $\sigma$  and corresponding chain  $\tau$  of Brauer pairs. If  $N_P(\sigma)$  is a proper subgroup of  $P$ , then  $f([\sigma]) = 0$ , and if  $N_P(\sigma) = P$ , then  $f([\sigma])$  is the number of height zero characters of the block  $e_\tau$  of  $N_G(\tau)$ . Dade's ordinary conjecture for  $B$ , reformulated here in terms of chains of Brauer pairs, asserts that  $\mathbf{k}_d(G, B)$  is equal to the alternating sum

$$\sum_{[\sigma] \in S_{\triangleleft}(\mathcal{C})} (-1)^{|\sigma|} f([\sigma]).$$

The passage between formulations in terms of normalisers of chains of Brauer pairs rather than normalisers of chains of  $p$ -subgroups is well known; see e.g. [6, 4.5], [15].

If  $|n(\sigma)| = |\sigma| + 1$ , then, setting  $H = N_G(\tau)$ , we have  $N_G(n(\tau)) = N_H(N_P(\tau), e_{n(\tau)})$ ; that is,  $(N_P(\tau), e_{n(\tau)})$  is a maximal  $(H, e_\tau)$ -Brauer pair. By the assumptions, the Alperin–McKay conjecture holds for the block  $e_\tau$  of  $H$ . This translates to the equality  $f([\sigma]) = f([n(\sigma)])$ . That is, the function  $f$  satisfies the hypotheses of Lemma 6. Thus the above alternating sum is equal to  $f([P])$ , which by definition is  $\mathbf{k}_d(N_G(P, e), e)$ , and thus the Alperin–McKay conjecture holds for  $B$ .  $\square$

Theorem 1 follows now immediately from combining Propositions 5 and 7.

**Remark 8.** By work of Dade [2] and Okuyama and Wajima [12], the Alperin–McKay conjecture holds for blocks of finite  $p$ -solvable groups. G. R. Robinson pointed out that Proposition 5 yields another short proof of this fact.

**Remark 9.** Let  $G$  be a finite group,  $B$  a block algebra of  $\mathcal{O}G$ ,  $(P, e_P)$  a maximal  $(G, B)$ -Brauer pair with associated fusion system  $\mathcal{F}$  on  $P$ , and let  $Z$  be a central  $p$ -subgroup of  $G$ . Let  $\eta$  be a linear character of  $Z$ , and suppose that  $\eta$  extends to a linear character  $\hat{\eta}$  of  $P$  satisfying  $\text{foc}(\mathcal{F}) \leq \text{Ker}(\hat{\eta})$ . The proof of

Lemma 4 is based on the fact that the  $*$ -construction  $\chi \mapsto \hat{\eta} * \chi$  yields a bijection  $\text{Irr}(B|1_Z) \rightarrow \text{Irr}(B|\eta)$ . There is some slightly more structural background to this. For  $\chi \in \text{Irr}(B)$ , denote by  $e(\chi)$  the corresponding central primitive idempotent in  $K \otimes_{\mathcal{O}} B$ . Set

$$e_1 = \sum_{\chi \in \text{Irr}_0(B|1_Z)} e(\chi), \quad e_{\eta} = \sum_{\chi \in \text{Irr}_0(B|\eta)} e(\chi).$$

Identify  $B$  with its image in  $K \otimes_{\mathcal{O}} B$ . Multiplying  $B$  by the central idempotents  $e_1$  and  $e_{\eta}$  in  $K \otimes_{\mathcal{O}} B$  yields the two  $\mathcal{O}$ -free  $\mathcal{O}$ -algebra quotients  $Be_1$  and  $Be_{\eta}$  of  $B$ . By [8, Theorem 1.1], there is an  $\mathcal{O}$ -algebra automorphism  $\alpha$  of  $B$  which induces the identity on  $k \otimes_{\mathcal{O}} B$  and which acts on  $\text{Irr}(B)$  as the map  $\chi \rightarrow \hat{\eta} * \chi$ . Thus the extension of  $\alpha$  to  $K \otimes_{\mathcal{O}} B$  sends  $e_1$  to  $e_{\eta}$ , and hence induces an  $\mathcal{O}$ -algebra isomorphism

$$Be_1 \cong Be_{\eta}.$$

We conclude this note with an observation regarding canonical height zero characters in nilpotent blocks, based in part on some of the above methods.

Let  $G$  be a finite group,  $B$  a block algebra of  $\mathcal{O}G$ ,  $P$  a defect group of  $B$ , and  $i \in B^P$  a source idempotent of  $B$ . Denote by  $\mathcal{F}$  the fusion system of  $B$  on  $P$  determined by the choice of  $i$ . Suppose that  $K$  is a splitting field for all subgroups of  $G$ . For  $V$  a finitely generated  $\mathcal{O}$ -free  $B$ -module, denote by

$$\Delta_{V,P,i} : P \rightarrow \mathcal{O}^{\times}$$

the map sending  $u \in P$  to the determinant of the  $\mathcal{O}$ -linear automorphism of  $iV$  induced by the action of  $u$  on  $V$  (this makes sense since all elements in  $P$  commute with  $i$ ). By standard properties of determinants, this map depends only on the  $(B^P)^{\times}$ -conjugacy class of  $i$  and the isomorphism class of the  $K \otimes_{\mathcal{O}} B$ -module  $K \otimes_{\mathcal{O}} V$ . Thus if  $V$  affords a character  $\chi \in \text{Irr}(B)$ , we write  $\Delta_{\chi,P,i}$  instead of  $\Delta_{V,P,i}$ .

**Proposition 10.** *With the notation above, let  $\chi \in \text{Irr}(B)$  and  $\eta \in \text{Irr}(P/\text{foc}(P))$ . Regard  $\eta$  as a linear character of  $P$ . We have*

$$\Delta_{\eta * \chi, P, i} = \eta^{\chi(i)} \Delta_{\chi, P, i}.$$

*Proof.* The statement makes sense as the value of  $\chi$  on an idempotent is a positive integer. Let  $V$  be an  $\mathcal{O}$ -free  $\mathcal{O}G$ -module affording  $\chi$ . By [8, Theorem 1.1] there exists an  $\mathcal{O}$ -algebra automorphism  $\alpha$  of  $B$  such that the module  $V^{\alpha}$  (obtained from twisting  $V$  by  $\alpha$ ) affords  $\eta * \chi$  and such that  $\alpha(ui) = \eta(u)ui$  for all  $u \in P$ . Since in particular  $\alpha(i) = i$ , it follows that

$$\Delta_{V^{\alpha}, P, i}(u) = \Delta_{V, P, i}(\eta(u)u)$$

for all  $u \in P$ . The result follows as  $\text{rank}_{\mathcal{O}}(iV) = \chi(i)$ .  $\square$

Denote by  $\text{Irr}'(B)$  the set of all  $\chi \in \text{Irr}(B)$  such that  $\Delta_{\chi, P, i}$  is the trivial map (sending all elements in  $P$  to 1). Set  $\text{Irr}'_0(B) = \text{Irr}'(B) \cap \text{Irr}_0(B)$ . The maximal local pointed groups on  $B$  are  $G$ -conjugate. Thus if  $P'$  is any other defect group of  $B$  and  $i' \in B^{P'}$  a source idempotent, then there exist  $g \in G$  and  $c \in (B^{P'})^{\times}$  such that  $P' = gPg^{-1}$  and  $i' = c g i g^{-1} c^{-1}$ . Therefore the map

$\Delta_{V,P,i}$  is trivial if and only if the map  $\Delta_{V,P',i'}$  is trivial, and hence the sets  $\text{Irr}'(B)$  and  $\text{Irr}'_0(B)$  are independent of the choice of  $P$  and  $i$ . The following is immediate.

**Proposition 11.** *The sets  $\text{Irr}'(B)$  and  $\text{Irr}'_0(B)$  are invariant under any automorphism of  $G$  which stabilises  $B$ .*

The next result shows that if  $B$  is nilpotent, then  $\text{Irr}'_0(B)$  consists of a single element.

**Proposition 12.** *Suppose that  $B$  is nilpotent. Then  $|\text{Irr}'_0(B)| = 1$ . Moreover, if  $p$  is odd, then the unique element of  $\text{Irr}'_0(B)$  is the unique  $p$ -rational height zero character in  $B$ .*

*Proof.* Let  $\chi \in \text{Irr}_0(B)$ . Since  $i$  is a source idempotent of  $B$ ,  $\chi(i)$  is prime to  $p$ . Hence if  $\eta, \zeta$  are linear characters of  $P$ , then  $\eta^{\chi(i)} = \zeta^{\chi(i)}$  implies that  $\eta = \zeta$ . Since  $B$  is nilpotent, we have that  $\text{foc}(\mathcal{F}) = [P, P]$  and  $|\text{Irr}_0(B)| = |P : [P, P]|$ . Thus, by Proposition 10, the map  $\chi \mapsto \Delta_{\chi,P,i}$  is a bijection from  $\text{Irr}_0(B)$  to  $\text{Irr}(P/[P, P])$ . This proves the first assertion.

Suppose that  $p$  is odd. Let  $\chi_0$  be the unique  $p$ -rational character in  $\text{Irr}_0(B)$ . Let  $W(k)$  be the ring of Witt vectors in  $\mathcal{O}$ . By the structure theory of nilpotent blocks (see [14]), there exists a  $W(k)G$ -module  $V$  affording  $\chi_0$ . Since the source idempotent  $i$  can be chosen to be in  $W(k)G$ , we have that  $\Delta_{\chi,P,i}$  takes values in  $W(k)$ . Since  $p$  is odd, it follows that the trivial character of  $P$  is the unique linear character of  $P$  which takes values in  $W(k)$ .  $\square$

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